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THE STABILITY OF PERIODIC REVERSIBLE SYSTEMS†

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Some results established in [1–4] for autonomous reversible systems are extended to periodic systems that are reversible under the substitution $t \to -t$, $\mathbf{x} \to \mathbf{M}(-t)\mathbf{x}$, $\mathbf{M}^2(t) \equiv \mathbf{E}$ (where $\mathbf{E}$ is the identity matrix), both in the non-resonant case and in the case of internal resonance. A complete solution of the problem of stability in the first approximation is derived for single-frequency resonances, which have no analogue in autonomous systems. It turns out that a system with one degree of freedom is either unstable or stable to any finite order. It is shown that the known conditions [5] do in fact guarantee formal stability in the rolling of a heavy, homogeneous, almost symmetric ellipsoid in the principal plane.

1. PRELIMINARY REMARKS

CONSIDER the system

$$\mathbf{x}^{\circ} = \mathbf{A}(t)\mathbf{x} + \mathbf{X}(\mathbf{x}, t); \quad \mathbf{x}, \mathbf{X} \in \mathbf{R}^k; \quad \mathbf{X}(0, t) \equiv 0 \tag{1.1}$$

where the left-hand sides are analytic in $\mathbf{x}$. The real $k \times k$ matrix $\mathbf{A}(t)$ and non-linear (in $\mathbf{x}$) vector-valued function $\mathbf{X}(\mathbf{x}, t)$ are assumed to be ω-periodic, continuous functions of the independent variable t, with piecewise continuous derivatives, so that they are represented by their Fourier series.

System (1.1) is said to be reversible with a linear automorphism $\mathbf{M}(t)$ if it is invariant under the substitution $t \to -t$, $\mathbf{x} \to \mathbf{M}(-t)\mathbf{x}$, where $\mathbf{M}(t)$ is some $k \times k$ matrix that may depend on t.

In this case, it is obviously true that

$$\frac{d\mathbf{M}}{d\tau} = \mathbf{M}(\tau)\mathbf{A}(-\tau) + \mathbf{A}(\tau)\mathbf{M}(\tau), \quad \mathbf{M}(\tau)\mathbf{X}(\mathbf{x}, -\tau) + \mathbf{X}(\mathbf{M}(\tau)\mathbf{x}, \tau) \equiv 0 \tag{1.2}$$

($\tau = -t$). This definition of a reversible system, which differs from the usual one (see, for example, [6]) only in that $\mathbf{M}$ is allowed to depend on t, directly implies that the reversibility property is invariant under any linear transformation $\mathbf{x} = \mathbf{B}(t)\mathbf{y}$, $\det \mathbf{B}(t) \geqslant \epsilon > 0$. In the new variables, the system

$$\mathbf{y}^{\circ} = \mathbf{A}_1(t)\mathbf{y} + \mathbf{Y}(\mathbf{y}, t) \tag{1.3}$$

admits of an automorphism $\mathbf{M}_1(t) = \mathbf{B}^{-1}(t)\mathbf{M}(t)\mathbf{B}(t)$.

System (1.1) is reducible to (1.3) (see [7]) with a constant matrix $\mathbf{A}_1$. If $\mathbf{M}^2 = \mathbf{E}$, it is also true

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that $\mathbf{M}_1^2 = \mathbf{E}$, and the relationships

$$\frac{d\mathbf{M}_1}{d\tau} = \mathbf{M}_1(\tau)\mathbf{A}_1 + \mathbf{A}_1\mathbf{M}_1(\tau),\ \mathbf{M}_1\frac{d\mathbf{M}_1}{d\tau} + \frac{d\mathbf{M}_1}{d\tau}\mathbf{M}_1 = 0$$

together with the fact that $\mathbf{A}_1$ is constant, imply that $\mathbf{M}_1^{\cdot} \equiv \mathbf{0}$. Consequently, $\mathbf{M}_1$ is also a constant matrix. This is the case considered below.

Let l_+ and l_- denote the number of eigenvalues of $\mathbf{M}_1$ equal to +1 and −1, respectively. Then there is a non-degenerate linear transformation that will reduce system (1.3) to a form in which

$$\mathbf{M}_1 = \begin{Vmatrix} \mathbf{E}_{l_+} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_{l_-} \end{Vmatrix}$$

($\mathbf{E}_j$ is the identity j-matrix).

Finally, we note that a normalizing transformation will preserve the constant linear automorphism $\mathbf{M}_1$ [8],† and combining the aforementioned together with known results [2, 8], we at once infer the following properties of system (1.1) with an automorphism $\mathbf{M}(t)$, $\mathbf{M}^2(t) \equiv \mathbf{E}$.

1. The system cannot be asymptotically stable; this follows from the fact that, apart from $\mathbf{y}(t)$, system (1.3) also has the solution $\mathbf{M}_1\mathbf{y}(-t)$.

2. The system may be stable only in the critical case in which it has m zero and n pairs of purely imaginary characteristic exponents.

3. If $l_+ - l_- = m$ and $l_- = n$, the system will be formally stable in the non-resonant case.

If $l_+ \geqslant l_-$, we have an N-system [2]. Obviously, besides property 3, the other properties of N-systems established in [2] are also valid. In the opposite situation, when $l_+ < l_-$, one can already establish instability [4, 9] with respect to second-order forms. We also note that the number of zero eigenvalues of a matrix $\mathbf{A}_1$ with simple elementary divisors is at least $|l_+ - l_-|$.

Suppose that in an N-system ($l_+ \geqslant l_-$) there are exactly $n = l_-$ pairs of purely imaginary eigenvalues $\pm\lambda_s$ that satisfy a resonance condition

$$p_1\lambda_1 + \ldots + p_n\lambda_n = i2\pi\omega^{-1}q,\ q \in Z, p_j \in Z\ (j = 1,\ldots,n) \tag{1.4}$$

where $|p_1| + \ldots + |p_n| \geqslant 1$ and identical values of λ_s have simple elementary divisors. Then system (1.3) may be written in the form

$$\begin{aligned} &\xi^{\cdot} = \Xi(\xi,\eta,\bar\eta,t) \\ &\eta^{\cdot} = \Lambda\eta + \mathrm{H}(\xi,\eta,\bar\eta,t),\ \bar\eta^{\cdot} = \Lambda\bar\eta + \bar{\mathrm{H}}(\xi,\eta,\bar\eta,t);\ \Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_n) \end{aligned} \tag{1.5}$$

(the bar denotes complex conjugation, ξ is a real m-vector and η and $\bar\eta$ are complex n-vectors) with a linear automorphism $t \to -t$, $\eta \to \bar\eta$, $\bar\eta \to \eta$. By virtue of this automorphism, the expansion of the non-linear terms Ξ, H, $\bar{\mathrm{H}}$ in powers of ξ, η, $\bar\eta$ with coefficients that are ω-periodic functions of t, of the form $c_\nu \exp(i2\pi/\omega\nu t)$, $\nu \in Z$, will contain only purely imaginary constants c_ν. This property is preserved under normalization (1.5).

In the case of multiple-frequency resonance, when the vector $\mathbf{p} = (p_1, \ldots, p_n)$ has at least two non-zero components, the problem of stability with respect to finite-order forms may be reduced completely to the autonomous case with $q = 0$ [10]. As the resonance coefficients are purely imaginary, the results of [1–3] hold in this case also.

A unique feature of periodic systems is the possibility of single-frequency resonance, when $q \neq 0$ in (1.4) and all components of $\mathbf{p}$ vanish except, say, p_1. Then a system of general form with odd-order (first or third) resonance is usually unstable [11]. This is also true of a reversible system (1.1).

†See also BRYUNO A. D., Sets of analyticity of a normalizing transformation. Preprint No. 97, Inst. Prikl. Mat., Moscow, 1974.

The problem of stability for odd p_1 has received fairly thorough treatment for Hamiltonian systems [12]. It turns out that the problem may also be solved quite exhaustively for reversible systems (1.1) to the first non-linear approximation, but in systems of general form it has been possible hitherto to establish only sufficient conditions for instability and asymptotic stability [13].

2. TWO AUXILIARY LEMMAS

The stability of the trivial solution of an ω-periodic reversible system (1.1) usually has to be considered when one is investigating a local neighbourhood of an ω-periodic motion of an autonomous reversible system of the type envisaged in the Heinbockel–Struble theorem [14]. We will state a generalization of that theorem, which will be needed later.

Lemma 1. Any motion $\varphi(t)$ of a reversible system

$$\mathbf{x}^{\cdot} = \mathbf{X}(\mathbf{x}, t), \mathbf{M}\mathbf{X}(\mathbf{x}, t) + \mathbf{X}(\mathbf{M}\mathbf{x}, -t) \equiv \mathbf{0}, \mathbf{x} \in \mathbf{R}^k, \det\mathbf{M} \neq 0 \qquad (2.1)$$

in which the right-hand side is 2π-periodic with respect to $x_1, \ldots, x_p$ $(p \leq k)$ and ω-periodic with respect to t, such that $\varphi(v\omega)$, $\varphi(v\omega + \Delta) \in \mathbf{L}$, $2\Delta = \omega l$, $v \in \mathbf{Z}$, $l \in \mathbf{N}$, is ωl-periodic in $\mathbf{X}^* = \mathbf{T}^p \times \mathbf{R}^{k-p}$ (where $\mathbf{T}^p$ is a p-dimensional torus in $x_1, \ldots, x_p$ space) if

$$\mathbf{L} = \{\mathbf{x} . \mathbf{x} = \mathbf{a}, \mathbf{M}^{2s+1}\mathbf{a} = \mathbf{a} + 2\pi\mathbf{q};\ s \in \mathbf{Z}\}$$

where $\mathbf{q} = (q_1, \ldots, q_k)$ is a vector with integer coefficients and $q_{p+1} = \ldots = q_k = 0$.

The question now arises, which motions $\varphi(t)$ of a reversible system (2.1) have neighbourhoods in which the equations of the perturbed motion are also reversible? The answer is given by the following lemma.

Lemma 2. Any motion $\varphi(t)$ such that $\varphi(\omega l) \in \mathbf{I}$ for some $l \in \mathbf{Z}$ has a neighbourhood in which the equations of the perturbed motion are reversible with an automorphism $\mathbf{M}^{2s+1}$.

To prove Lemma 2, we will assume without loss of generality that $\varphi(0) \in \mathbf{L}$ and consider the equation of perturbed motion

$$\mathbf{y}^{\cdot} = \mathbf{Y}(\mathbf{y}, t) \equiv \mathbf{X}(\mathbf{y} + \varphi(t), t) - \mathbf{X}(\varphi(t), t)$$

We have $\mathbf{M}^{2s+1}\varphi(0) = \varphi(0) + 2\pi\mathbf{q}$ for some $\mathbf{q}$. Thus, on the assumption that the solution is unique, $\mathbf{M}^{2s+1}\varphi(t) = \varphi(-t) + 2\pi\mathbf{q}$. Hence, since $\mathbf{X}(\mathbf{x}, t)$ is periodic in $x_1, \ldots, x_p$, we obtain

$$\mathbf{M}^{2s+1}\mathbf{X}(\mathbf{y} + \varphi(t), t) = -\mathbf{X}(\mathbf{M}^{2s+1}(\mathbf{y} + \varphi(t)), -t) = -\mathbf{X}(\mathbf{M}^{2s+1}\mathbf{y} + \varphi(-t), -t)$$

$$\mathbf{M}^{2s+1}\mathbf{X}(\varphi(t), t) = -\mathbf{X}(\mathbf{M}^{2s+1}\varphi(t), -t) = -\mathbf{X}(\varphi(-t), -t)$$

that is

$$\mathbf{M}^{2s+1}\mathbf{Y}(\mathbf{y}, t) + \mathbf{Y}(\mathbf{M}^{2s+1}\mathbf{y}, -t) = \mathbf{0}$$

3. A SYSTEM WITH ONE DEGREE OF FREEDOM

Consider the case of a system with one degree of freedom, when $m = 0$, $n = 1$.

Fourth-order resonance. $4\lambda_1 = i2\pi\omega^{-1}q$. Normalize system (1.5) to Kth order terms inclusive.

Then, using the previous notation, we have

$$\eta_1^{\cdot}=\lambda_1\eta_1+i\eta_1 \sum_{\substack{\alpha\geq 1\\ 3\leq 2\alpha+4\beta+1\leq K}} C_{\alpha\beta}(\eta_1\bar{\eta}_1)^{\alpha}(\bar{\eta}_1 e^{\lambda_1 t})^{4\beta}+\dots \quad (3.1)$$

(the unwritten terms are of order greater than K with respect to η_1 and $\bar{\eta}_1$, C are real constants, and the complex-conjugate equation is omitted). If we make the replacement of variable $\eta_1=z_1e^{\lambda_1 t}$ in (3.1) we obtain the following equation in variables z_1 and $\bar{z}_1$

$$z_1^{\cdot}=iz_1 \sum_{\substack{\alpha\geq -1\\ 3\leq 2\alpha+4\beta+1\leq K}} C_{\alpha\beta}(z_1\bar{z}_1)^{\alpha}\ \bar{z}_1^{4\beta}+\dots$$

Finally, changing to polar coordinates $z_1=r_1^{1/2}e^{i\theta_1}$, we get

$$r_1^{\cdot}=2C_{-1,1}r_1^2\sin\theta+2\sum_{\substack{\alpha\geq -1\\ 5\leq 2\alpha+4\beta+1\leq K}} C_{\alpha\beta}r_1^{\alpha+2\beta+1}\sin\beta\theta+\dots;\ \theta=4\theta_1 \quad (3.2)$$

$$\theta^{\cdot}=2(C_{1,0}+C_{-1,1}\cos\theta)r_1+2\sum_{\substack{\alpha\geq -1\\ 4\leq 2\alpha+4\beta+1\leq K-1}} C_{\alpha\beta}r_1^{\alpha+2\beta}\cos\beta\theta+\dots$$

Hence, it is obvious that if $|C_{1,0}|<|C_{-1,1}|$ the model system, truncated at the first non-linear terms, has an increasing solution which is a ray, implying instability with respect to third-order forms.

Let $|C_{1,0}|>|C_{-1,1}|$. Then it is obvious from the second equation of (3.2) that, for sufficiently small r_1, θ is a monotone function of time.

The system is reversible with an automorphism $t\to -t$, $r_1\to r_1$, $\theta\to-\theta$. The set **L** of Lemma 1 for (3.1) is the pair of rays $\theta=0$, π. Therefore, as the angle θ varies monotonically, any trajectory of the autonomous system obtained from (3.2) by dropping the unwritten terms (i.e. truncated at the Kth order terms, where K is any finite integer) will cut the rays $\theta=0$, $\theta=\pi$ and will be a periodic motion.

Theorem 1. Suppose that $|C_{1,0}|<|C_{-1,1}|$ for a system with one degree of freedom and a fourth-order single-frequency resonance. Then the system is unstable in Lyapunov's sense. If the reverse inequality is true, the system is stable to any finite order.

Second-order resonance. $2\lambda_1=i2\pi\omega^{-1}$. A similar analysis also holds in this case. The analogue of (3.2) is the system

$$\frac{1}{2}r_1^{\cdot}=(C_-\sin\theta+C_{-1,2}\sin2\theta)r_1^2 \sum_{\substack{\alpha\geq -1\\ 5\leq 2\alpha+2\beta+1\leq K}} C_{\alpha\beta}r_1^{\alpha+\beta+1}\sin\beta\theta+\dots \quad (3.3)$$

$$\frac{1}{2}\theta^{\cdot}=(C_{1,0}+C_+\cos\theta+C_{-1,2}\cos2\theta)r_1+\sum_{\substack{\alpha\geq -1\\ 4\leq 2\alpha+2\beta+1\leq K-1}} C_{\alpha\beta}r_1^{\alpha+\beta}\cos\beta\theta+\dots$$

$$\theta=2\theta_1,\ C_{\pm}=C_{0,1}\pm C_{2,-1}$$

Theorem 2. Given a system with one degree of freedom and a second-order single-frequency resonance such that

$$D=C_+^2+8C_{-1,2}C_*\geq 0,\ |C_+|-D^{1/2}|<4|C_{-1,2}| \quad (3.4)$$

($C_*=C_{1,0}-C_{-1,2}$), the system is generally (that is, if none of the roots of the quadratic equation $2C_{-1,2}\kappa^2+C_+\kappa+C_*=0$ and the linear equation $2C_{-1,2}\kappa+C_-=0$ are equal) unstable in the

Lyapunov sense. If at least one of the inequalities is reversed, the system is stable to any finite order.

Using the truncated systems (3.2) and (3.3), one can carry out an elementary analysis in the phase plane of the model system obtained by considering only the first non-linear terms. For example, in the case of second-order resonance, the qualitative phase portraits in the complex plane, in all non-degenerate cases and a few degenerate cases, are those shown in Fig. 1 with the following values of the parameters

1. $|\kappa_{1,2}| > 1, \mathrm{Im}\kappa_{1,2} = 0$ or $\mathrm{Im}\kappa_{1,2} \neq 0$
2. $|\kappa_1| < 1, |\kappa_2| > 1, \mathrm{Im}\kappa_{1,2} = 0$
3. $|\kappa_{1,2}| < 1, \mathrm{Im}\kappa_{1,2} = 0, (2C_{-1,2}\kappa_1 + C_-)(2C_{-1,2}\kappa_2 + C_-) > 0$
4. $|\kappa_{1,2}| < 1, \mathrm{Im}\kappa_{1,2} = 0, (2C_{-1,2}\kappa_1 + C_-)(2C_{-1,2}\kappa_2 + C_-) < 0$
5. $\kappa_1 = 1, |\kappa_2| > 1$
6. $\kappa_1 = 1, |\kappa_2| < 1, (2C_{-1,2} + C_-)(2C_{-1,2}\kappa_2 + C_-) < 0$
7. $\kappa_1 = \kappa_2, |\kappa_{1,2}| < 1$
8. $|\kappa_1| < 1, |\kappa_2| > 1, 2C_{-1,2}\kappa_1 + C_- = 0$
9. $|\kappa_{1,2}| < 1, 2C_{-1,2}\kappa_1 + C_- = 0.$

The non-degenerate cases are those in which the roots $\kappa_{1,2}$ of the quadratic equation are distinct and neither of them is a root of the linear equation or ±1 (cases 1–4) In some situations (e.g. 2 and 4) the trajectories may either go off to infinity as $t \to \pm\infty$ (the solid curves in Fig. 1) or tend to zero (the dashed curves). The exact shape of each trajectory depends on the sign of the derivative with respect to θ of the coefficient of r_1 in the equation for θ on the ray (e.g. if the derivative becomes negative as one moves along the ray to infinity, the solution will go to infinity in the vicinity of the ray; otherwise it will tend to zero).

Generalization to $m \geq 1$, $n = 1$. When there are zero roots corresponding to the variables ξ the normal form (3.2) or (3.3) must also contain a differential equation for the vector ξ in which there are no terms of order up to K inclusive. The right-hand side of the equation for θ will contain an additional term, which vanishes at $\xi = 0$.

Clearly, the third-approximation system will have increasing solutions at $\xi = 0$ when the conditions for the existence of the latter, as ascertained for $m = 0$, are satisfied. Suppose that these conditions fail to hold. Since $\xi = \mathrm{const}$ are first integrals of the truncated system, it follows that, in an unstable system, r_1 will increase along the relevant trajectory. Therefore, if the trajectory connects points inside a δ-neighbourhood and outside an ϵ-neighbourhood, then $\|\xi\| < \delta$ on the trajectory and when $r_1^2 \sim \delta$ the angle θ will be determined by a term which depends only on r_1, that is, it will preserve its sign. Consequently, by Lemma 1, it will be a periodic solution. If this set of periodic solutions leads to instability, then, since δ is arbitrarily small, a periodic solution with an arbitrarily large period exists. But such a motion must also exist when $m = 0$, and this is impossible by assumption.

Theorem 3. Theorems 1 and 2 remain valid for any $m > 0$.

4. MULTIDIMENSIONAL SYSTEMS

Suppose that system (1.5) has a fourth-order single-frequency resonance. Then, after normalization, the model system of the first non-linear approximation will be

$$
\begin{aligned}
&r_1^{\cdot} = 2C_{-1,1} r_1^2 \sin\theta \\
&\theta^{\cdot} = 2\left[\alpha(\xi) + \sum_{\mu=2}^{n} a_\mu r_\mu\right] + 2(C_{1,0} + C_{-1,1}\cos\theta) r_1 \\
&r_s^{\cdot} = 0, \ \xi_j^{\cdot} = 0 \ (s = 2, \dots, n; \ j = 1, \dots, m)
\end{aligned}
\tag{4.1}
$$

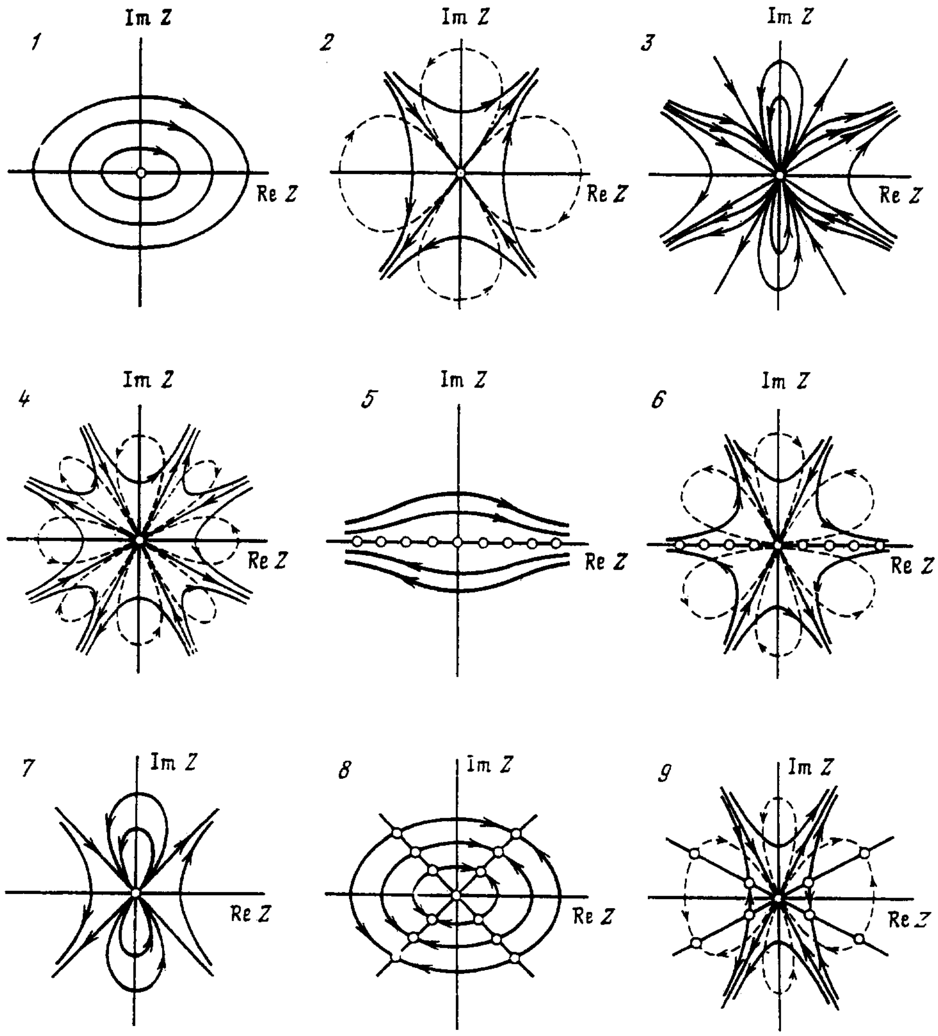

FIG. 1.

($\alpha(\xi)$ is a quadratic polynomial in ξ, without free term). Obviously, if $|C_{1,0}|<|C_{-1,1}|$ system (4.1), like the original system, will be unstable, since under this condition it has an increasing solution—a ray—on the manifold $r_s=0$, $\xi_j=0$ $(s=2,\ldots,n;\ j=1,\ldots,m)$. If the reverse inequality is true, system (4.1) will be stable, because the integral

$$\Phi=(C_{1,0}+C_{-1,1}\cos\theta)r_1^2+r_1[\alpha(\xi)+\sum_{\mu=2}^{n}a_\mu r_\mu]+\gamma(\sum_{j=2}^{m}\xi_j^2+\sum_{\mu=2}^{n}r_\mu^2)(\gamma-\text{const})$$

is sign definite.

Theorem 4. A necessary and sufficient condition (up to the equality sign) for stability of a model system with fourth-order single-frequency resonance is that $|C_{1,0}|<|C_{-1,1}|$.

The analysis of stability for second-order resonance is much more difficult. The model system is then

$$\xi_j^{\cdot}=0\quad(j=1,\ldots,m)$$

$$\frac{1}{2}r_1^{\cdot}=[\beta(\xi)+\sum_{\mu=2}^{n}b_\mu r_\mu+(C_-+2C_{-1,2}\cos\theta)r_1]r_1\sin\theta \tag{4.2}$$

$$\frac{1}{2} r_s^{\cdot} = C_s r_s r_1 \sin\theta \ (s = 2,\ldots,n)$$

$$\frac{1}{2}\theta^{\cdot} = \alpha(\xi) + \beta(\xi)\cos\theta + \sum_{\mu=2}^{n}(a_\mu + b_\mu \cos\theta) r_\mu + (C_{1,0} + C_{+}\cos\theta + C_{-1,2}\cos 2\theta) r_1$$

($\alpha(\xi)$ and $\beta(\xi)$ are quadratic polynomials in ξ without free terms and a_μ, b_μ and C_s are real constants). This system admits of a linear automorphism $t \to -t$, $\xi \to \xi$, $\mathbf{r} \to \mathbf{r}$, $\theta \to -\theta$.

Clearly, if condition (3.4) holds, the system is unstable, and there will be increasing solutions in the form of rays. Suppose the condition is false. It is obvious from (4.2) that each of the manifolds $r_j = 0$ $(j = 1, \ldots, n)$ is invariant; moreover, $r_1 = 0$ consists solely of steady and periodic motions of the form $r_s = \text{const}$, $\xi_j = \text{const}$ $(s = 2, \ldots, n;\ j = 1, \ldots, m)$. In addition, it was shown in Sec. 3 that there are no solutions on the manifold $r_s = 0$ $(s = 2, \ldots, n)$ that produce instability. It follows that on such trajectories necessarily $r_1 r_\alpha \neq 0$ for at least one $\alpha = 2, \ldots, n$.

It turns out that on each of the manifolds

$$W_\alpha = \{\xi, \mathbf{r} : \xi = 0, r_j = 0 \ (j = 2,\ldots,n,\ j \neq \alpha),\ r_1 r_\alpha \neq 0\}$$

system (4.2) may have an increasing solution in the form of a ray. A sufficient condition for this to be true is that the system

$$\begin{aligned} &b_\alpha \kappa_\alpha + (C_{-} - C_\alpha + 2C_{-1,2}\cos\theta_*)\kappa_1 = 0 \\ &(a_\alpha + b_\alpha \cos\theta_*)\kappa_\alpha + (C_{1,0} + C_{+}\cos\theta_* + C_{-1,2}\cos 2\theta_*)\kappa_1 = 0 \end{aligned} \tag{4.3}$$

should have a positive solution κ_1, κ_α for which $\sin\theta_* \neq 0$. Obviously, if system (4.3) has such a solution at $+\theta_*$, it will have a solution at $-\theta_*$ as well. Thus, for each solution κ_1, κ_α of system (4.3) there are two rays $\theta = \pm\theta_*$ such that the solution goes to infinity along one $(C_\alpha \sin\theta_* > 0)$ and tends asymptotically to zero along the other. It is not hard to derive conditions on the coefficients of the system for this to be true.

Let us consider a case in which system (4.3) is not consistent for any α, i.e. the manifolds W_α do not contain solutions that are rays. All the trajectories on the integral manifold

$$V_k = \{\xi, \mathbf{r} : r_1 \neq 0, \ldots, r_k \neq 0,\ r_{k+1} = \ldots = r_n = 0 \ (1 \leqslant k \leqslant n)\}$$

may be divided into three types: (1) sin θ vanishes at least twice on the trajectory; (2) sin θ vanishes only once (at $t = t_*$); (3) $\sin\theta \neq 0$ on the trajectory. In the first case, by Lemma 1, the solution is periodic. Case 2 when $t > t_*$ (or $t < t_*$) reduces to case 3.

Let us consider case 3, noting in advance that the variable r_μ for which $C_\mu = 0$ may be associated with the variable ξ. If there is a sign change in the sequence of numbers $C_2, \ldots, C_k$ and moreover

$$J = \int_0^T r_1(t) \sin\theta(t)\, dt = \infty \tag{4.4}$$

then the solution cannot be bounded; here T is the time the solution exists. But if $C_2, \ldots, C_k$ are all of the same sign, then either the solution is unbounded $(JC_s = +\infty)$ or it asymptotically approaches a steady value $\xi = \text{const}(JC_s = -\infty)$. If J is finite, the solution will be asymptotic to a steady solution on the set $\theta^{\cdot} = 0$. When that happens, necessarily $r_1 \sin\theta \to 0$, and as $r_1 \to 0$ we must have $r_s \to \infty$ for at least one of $s = 2, \ldots, k$, which is impossible by assumption. Consequently, $\sin\theta \to 0$, and the steady value does not belong to the family $\xi = \text{const}$, $\mathbf{r} \to \mathbf{0}$. Otherwise, the reversibility property of (4.2) implies that there is a periodic motion and J does not exist.

We will show that, if there are no ray solutions, the system is stable in Lyapunov's sense. System (4.2) is homogeneous up to a transformation of the inessential variables ξ. Stability will therefore follow if we can show that the set of all trajectories that begin in a certain δ_*-sphere S_* is bounded.

Suppose first that the numbers $C_2, \ldots, C_k$ are all distinct; we may assume without loss of generality that $0 < C_2 < C_3 < \ldots < C_k$. The equations for r_s $(s = 2, \ldots, n)$ imply that for any $K > 1$, if $r_i > Kr_i^*$, then also $r_j = K^{C_j/C_i} r_j^*$ $(i, j = 1, \ldots, k)$, where $(\xi^*, \mathbf{r}^*) \in S_*$.

For any $v > 0$, therefore, $r_j < vr_i$, $i \neq j$, if $r_j^* < vK^{1-C_j/C_i} r_j^*$.

Let $i = 2$, $j \neq 2$. For sufficiently small $v = v_2$ and arbitrary $K = K_2$, the solutions such that $(\xi^* \mathbf{r}^*) \in \mathbf{U}_2 = \{\xi^*, \mathbf{r}^*: \mathbf{r}_j^* < v_2 K_2^{1-C_j/C_2} r_2^*;\ j = 3, \ldots, k\}$ satisfy the condition $r_j < v_2 r_2$ $(j = 3, \ldots, k)$, and the sign of $\theta^\cdot$ on the corresponding trajectories (except possibly for a small neighbourhood of $\theta^\cdot = 0$) is determined by the terms with r_1 and r_2. Fix some K_2.

For arbitrary $v_3 > 0$, $K_3 \geqslant 1$, define a set

$$\mathrm{U}_3 = \{\xi^*, \mathbf{r}^*: r_3^* \geqslant v_2 K_2^{1-C_3/C_2} r_2^*,\ \mathbf{r}_j^* < v_3 K_3^{1-C_j/C_3} r_3^*;\ j = 4, \ldots, k\}$$

For sufficiently small v_3 and arbitrary K_3, the sign of $\theta^\cdot$ for a solution with $(\xi^*, \mathbf{r}^*) \in \mathbf{U}_3$ is determined by r_1, r_2 and r_3, but for sufficiently large $K_3 > K_2$ it depends only on r_1 and r_3, provided that $K_2^* r_3^* \leqslant r_3 \leqslant K_3 r_3^*$ (where $K_2 < K_2^*$ is some number). This follows from the condition $C_3 > C_2$ and the equations for r_2 and r_3. Continuing the process, we divide S_* into sets U_s $(s = 2, \ldots, k)$ such that if $(\xi^*, \mathbf{r}^*) \in \mathbf{U}_s$ and $K_{s-1}^* r_s^* \leqslant r_s \leqslant K_s r_s^*$, the sign of $\theta^\cdot$ depends on the terms with r_1 and r_s only, where the numbers K_{s-1}^*, K_s are independent of the specific choice of a point in U_s.

Looking at a solution with $(\xi^*, \mathbf{r}^*) \in \mathbf{U}_\alpha$ for $K_{\alpha-1}^* r_\alpha^* \leqslant r_\alpha \leqslant K_\alpha r_\alpha^*$, we see that there are two possibilities: (a) $|\theta^\cdot| > 2\epsilon r_1$, where ϵ is some positive number; (b) the solution is in the set $|\theta^\cdot| \leqslant 2\epsilon r_1$. In the first case, estimating $dr_j/d\theta$ $(j = 1, \ldots, k)$, we find that $r_\alpha^* \leqslant r_\alpha \exp[C_\alpha(\theta - \theta^*)/\epsilon]$ $(\theta^* = \text{const})$. Consequently, if K_α is sufficiently large, then $\sin\theta$ vanishes twice, and by Lemma 1 such motions in $\mathbf{U}_s$ are periodic.

Now, assuming that ϵ is sufficiently small, let us calculate the derivative of $\theta^\cdot$ in case (b). We have

$$\theta^{\cdot\cdot} = \varphi(\theta)\,[r_\alpha + g(r_1, \ldots, r_k, \xi_1, \ldots, \xi_m, \theta)]\,r_1 \sin\theta,\quad |g| \ll r_\alpha$$

where φ is a quadratic polynomial in $\cos\theta$. It can be shown that $\varphi(\theta)$ may vanish either when conditions hold that guarantee the existence of a ray, or in the degenerate cases ($\varphi(0) = 0$ or $\varphi(\pi) = 0$). Disregarding these cases and estimating $dr_j/d\theta^\cdot$ $(j = 1, \ldots, k)$, we conclude that the solution will either leave the domain $|\theta^\cdot| \leqslant \epsilon r_1$ (after which case (a) will hold) or reach a point of rest. Thus, all the solutions in $\mathbf{U}_\alpha$ are bounded.

Now suppose that some of the numbers $C_2, \ldots, C_n$ are equal, say these are $C_{\alpha_1}, \ldots, C_{\alpha_\gamma}$. Then system (4.2) may have ray solutions in the sets

$$\mathrm{W}_{\alpha_1 \ldots \alpha_\gamma} = \{\xi, \mathbf{r}: \xi = 0,\ r_j = 0\ (j = 2, \ldots, n;\ j \neq \alpha_i, i = 1, \ldots, \gamma)\}$$

In that case

$$\varphi(\theta) = \sum_{j=1}^{\gamma} [a_{\alpha_j}(C_{\alpha_j} - C_-) + b_{\alpha_j}(C_{1,0} - C_{-1,2}) + b_{\alpha_j}(C_{\alpha_j} + C_+ - C_-)\cos\theta]\,\kappa_{\alpha_j}$$

where κ_{α_j} are certain positive constants, and a ray $\theta = \theta^*$ exists if $\varphi(\theta_*) = 0$, $\sin\theta_* \neq 0$ and

$$\sum_{j=1}^{\gamma} (a_{\alpha_j} + b_{\alpha_j}\cos\theta_*)\,\kappa_{\alpha_j} + (C_{0,1} + C_+\cos\theta_* + C_{-1,2}\cos 2\theta_*) = 0 \tag{4.5}$$

If there are no rays, stability may be established in this case too.

Theorem 4. Suppose that the non-zero coefficients C_s in a system (4.2) fall into β groups of equal numbers $C_{\alpha_1}, \ldots, C_{\alpha_\gamma}$. Then a necessary and sufficient condition (up to equalities) for the stability of (4.2) is that at least one of the following conditions should hold in the non-negative cone $\kappa_{\alpha_j} \geqslant 0$ ($j=1, \ldots, \gamma$) for all β: (1) Eq. (4.5) has no roots, (2) for each root θ_* of Eq. (4.5), we have $\varphi(\theta_*) \neq 0$. In the unstable case system (4.2) has an increasing solution that is a ray, and the original solution is unstable with respect to third-order forms.

5. EXAMPLE

Let us analyse the stability of rolling of a heavy homogeneous ellipsoid in the principal section. As variables, we take x, y, z—the coordinates of the points of contact, and p, q, r—the projections of the angular velocity in a system of coordinates rigidly attached to the ellipsoid with axes parallel to the latter's axes. The equations of motion [15] are reversible [2] and admit of three linear automorphisms (one of the pairs (x, p), (y, q), (z, r) changes sign). For motion in the x, y plane we have $p=q=0$, $z=0$. If r vanishes twice, then by Lemma 1 the motion is periodic—oscillations about an equilibrium position. The second possibility is $r \neq 0$ [16]. Here x (or y) vanishes, and by Lemma 1 the motion is again periodic—rolling in one direction.

Let us assume from now on that the motion takes place without jumping; stability under such conditions has been analysed in the linear approximation.†

The set $\mathbf{L}$ of Lemma 2 consists of the points at which one of the above pairs vanishes. Hence the equations of the perturbed motion in the neighbourhood of rolling are again reversible with periodic right-hand sides. Replacing the variables x, y by generalized polar coordinates $x=a\rho\cos\varphi$, $y=b\rho\cos\varphi$ (where a, b and c are the semi-axes of the ellipsoid) and transforming to a new time variable φ, we see that to a first approximation the equations for p, q and z are separable and 2π-periodic with respect to φ.

By reversibility, this third-order linear system has a reciprocal characteristic equation [17, p. 71]. Hence one of its roots is one, and the other two are determined by the equation $\kappa^2-24\kappa+1=0$ and $|\kappa_{1,2}|=1$, if $|A|<1$. If the ellipsoid is nearly symmetric ($a=a_0(1-\epsilon)$, $b=a_0(1+\epsilon)$, $\epsilon \ll 1$), the number A, and hence also $\kappa_{1,2}$, may be found by expanding in powers of ϵ. It turns out that the condition of [5] applied to this ellipsoid, up to terms of order ϵ^2, guarantees $|A|<1$ if we define ω to be the mean angular velocity of rolling. Hence, the condition for stability is

$$\omega^2 > a_0 \frac{(a_0^2+c^2)(a_0-c)}{14a_0^4} g + a\epsilon^2 + \ldots \quad (g=9.81\ldots)$$

By property 3 in Sec. 1, if this condition holds, the rolling motion will be formally stable, provided there is no resonance.

Analogous expressions may be found for the roots $\kappa_{1,2}$.

Unfortunately, these expressions are all quite lengthy and cannot be reproduced here.

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